J80-165

Creep Linearization of Nonaxisymmetrically Heated Cylinders

30003
80004
80009

M. R. Eslami*
Tehran Polytechnic, Tehran, Iran

Introduction

THE creep phenomenon in engineering materials is becoming more important as the industry tends toward optimum production and the use of materials in higher working pressure and temperature environments. The importance of the creep theory is evident as it is substituted for the theory of elasticity in the design state of the mechanical components, because many of the elasticity rules do not apply for the materials in high temperature conditions. In these conditions elasticity is generally taken as the initial condition and the set of stresses and strains changes after loading. The stress distribution in the creep regime is governed by a set of nonlinear stress-strain relations which yield nonlinear partial differential equations. In the classical problems the solutions of these differential equations are obtained for some specific simple cases, but for more general conditions no general theoretical solution exists.

Among all classical problems, thick wall cylinders (because of their extensive use in industry) are most often discussed and analyzed; therefore more data are available concerning their creep stress distribution for different types of loadings. Most of these data and the methods of solution are for the case of symmetrical loadings. For a general case of loading, Nadai[1] has set up a governing equation for creep stress calculation in the cylinders, but no solution is presented. Rabotnov[2] has also discussed the general governing equations of creep and has presented a creep solution of a beam by linearizing the initial creep equation. Eslami and Sabbaghian[3] presented a simple numerical method for calculation of creep stresses in a thick-walled cylinder subjected to a nonaxisymmetric temperature field. The problem of steady creep in a nonuniformly heated thick cylinder subjected to radial temperature variation and internal pressure is discussed by Kachanov.[4] Rozenblyum[5] considers the effect of temperature variation over the length of the cylinder. Danyushevskii and Listvinskii[6] presented the solution of steady creep in a thick cylinder operating under internal pressure with a relatively small deviation of the temperature field from axisymmetric distribution. The temperature distribution is considered to be of a specific type.

This paper presents a series solution for a thick-walled cylinder subjected to internal pressure and general nonaxisymmetric temperature distribution, and with a relatively small deviation of the temperature field from axisymmetric distribution. The solution is based on linearizing the stress-strain relation. It can well be employed in analysis of other types of the mechanical members.

General Formulation

Consider a thick-walled cylinder of inner radius a and outer radius b, subjected to inside pressure P. The temperature does not vary along its axis, but is in general a function of r and ϕ, with the assumption that the temperature variation in the ϕ direction is relatively small. In general the temperature distribution can be shown as[3]:

$$\theta(r,\phi)=\theta_a+\theta_*\ln\frac{r}{a}+\lambda\sum_{n=1}^{\infty}\theta_n(r)\cos n\phi \tag{1}$$

where θ_a, θ_* are constants and $\theta_n(r)$ is a function of r. The parameter λ is assumed to be small relative to unity, such that its second, third, and higher order powers can be neglected.

Using the incompressibility condition and considering the plane strain condition $\epsilon_z=0$, the rate version of the compatibility equation in nondimensional form becomes[4]:

$$\frac{\partial^2\dot\epsilon_r}{\partial\phi^2}-3\rho\frac{\partial\dot\epsilon_r}{\partial\rho}-\rho^2\frac{\partial^2\dot\epsilon_r}{\partial\rho^2}=\frac{\partial^2}{\partial\rho\partial\phi}(\rho\dot\epsilon_{r\phi}) \tag{2}$$

where $\rho=r/b$.

The creep strain-stress relations for the plane strain condition are:

$$\dot\epsilon_r=\frac{3}{4}\frac{\dot\epsilon^*}{\sigma^*}(\sigma_r-\sigma_\phi),\quad \dot\epsilon_\phi=\frac{3}{4}\frac{\dot\epsilon^*}{\sigma^*}(\sigma_\phi-\sigma_r),\quad \dot\epsilon_{r\phi}=\frac{3}{2}\frac{\dot\epsilon^*}{\sigma^*}\tau_{r\phi} \tag{3}$$

where σ^* and $\dot\epsilon^*$ are the effective stress and the rate of effective strain, respectively.[1]

The stress distribution in the cylinder must satisfy Eqs. (2) and (3) along with the equilibrium equations and the boundary conditions. The stresses will satisfy the equilibrium equations if a stress function $\Phi(r,\phi)$ of the following form

$$\Phi(r,\phi)=\Phi_0(r)+\lambda\sum_{n=1}^{\infty}\Phi_n(r,\phi) \tag{4}$$

is introduced, such that the following relations exist

$$b^2\sigma_r^0=\frac{1}{\rho}\frac{\partial\Phi_0}{\partial\rho},\quad b^2\sigma_\phi^0=\frac{\partial^2\Phi_0}{\partial\rho^2} \tag{5}$$

$$b^2\sigma_r^n=\frac{1}{\rho^2}\frac{\partial^2\Phi_n}{\partial\phi^2}+\frac{1}{\rho}\frac{\partial\Phi_n}{\partial\rho},\quad b^2\sigma_\phi^n=\frac{\partial^2\Phi_n}{\partial\rho^2}$$

$$b^2\tau_{r\phi}^n=-\frac{\partial}{\partial\rho}\left[\frac{1}{\rho}\frac{\partial\Phi_n}{\partial\phi}\right] \tag{6}$$

where

$$\sigma_r=\sigma_r^0+\lambda\sum_{n=1}^{\infty}\sigma_r^n,\quad \sigma_\phi=\sigma_\phi^0+\lambda\sum_{n=1}^{\infty}\sigma_\phi^n,$$

$$\tau_{r\phi}=\lambda\sum_{n=1}^{\infty}\tau_{r\phi}^n \tag{7}$$

The superscript 0 indicates the axisymmetric solution and the superscript n indicates the "additional stresses" due to nonaxisymmetric temperature distribution.

The problem is to find Φ such that Eqs. (2) and (3) along with the boundary conditions are satisfied. In order to do so a constitutive law of creep of the following form may be considered

$$\dot\epsilon^*=B_0e^{c\theta}(\sigma^*)^m \tag{8}$$

where B_0, c, and m are constants and (*) denotes effective stress or strain. Based on this equation, the relations between rate of strains and stresses can be obtained. The effective stress for this case is[1]:

$$\sigma^*=[(\sigma_r-\sigma_\phi)^2+4\tau_{r\phi}^2]^{1/2} \tag{9}$$

By substituting Eqs. (9) and (1) into Eq. (3) the creep strain-stress relations will be fully determined, which are highly nonlinear. This set of relations can be linearized upon power expansion in λ, and the assumption of λ being small relative to unity so that its second and higher order powers can be neglected.

Received May 29, 1979; revision received Nov. 5, 1979.

Index categories: Thermal Stresses; Analytical and Numerical Methods; Structural Design.

*Associate Professor, Mechanical Engineering Dept. Member AIAA.

Expanding the creep coefficient $B_0 e^{c\theta}$ by power series, and neglecting the higher order terms, one would obtain

$$B_0 e^{c\theta} = B_0\left(1+\lambda c\sum_{n=1}^{\infty}\theta_n(r)\cos(n\phi)\right)\times\exp[c(\theta_a+\theta_*\ln\beta\rho)] \tag{10}$$

Substituting Eqs. (7) into Eq. (9), expanding the right-hand side of Eq. (9) by the binomial series, and neglecting the higher order terms yield

$$\sigma^* = \sigma_0^* + \lambda\sum_{n=1}^{\infty}\sigma_n^* \tag{11}$$

where $\sigma_0^* = \sigma_\phi^0 - \sigma_r^0$ and $\sigma_n^* = \sigma_\phi^n - \sigma_r^n$.

Solution for Axisymmetric Stresses

The axisymmetric stresses σ_r^0 and σ_ϕ^0 are the solutions of the cylinder subjected to axisymmetric temperature distribution $\theta_a + \theta_*\ln\beta\rho$ and the following boundary conditions

$$\sigma_r^0 = -p \text{ at } \rho = 1/\beta, \quad \sigma_r^0 = 0 \text{ at } \rho = 1 \tag{12}$$

The stresses obtained are[2]:

$$\sigma_r^0 = \frac{Ab^{-2K_1}}{2K_1}[1-\rho^{-2K_1}]$$

$$\sigma_\phi^0 = \frac{Ab^{-2K_1}}{2K_1}[(2K_1-1)\rho^{-2K_1}+1]$$

$$\sigma_0^* = \sigma_\phi^0 - \sigma_r^0 = K_1K_2\rho^{-2K_1} \tag{13}$$

where

$$A = \frac{2K_1 p}{a^{-2K_1} - b^{-2K_1}}$$

$$K_1 = \frac{1}{m}\left(1+\frac{c\theta_*}{2}\right), \quad K_2 = \frac{2p}{\beta^{2K_1}-1}$$

$$\beta = b/a \tag{14}$$

Solution for Nonaxisymmetric Stresses

Substituting Eqs. (13) into Eq. (11) and making use of Eq. (10), the creep strain-stress relations [Eqs. (3)] can be expanded by a power series in λ which, keeping the linear terms, leads to the following relations

$$\dot\epsilon_r = \dot\epsilon_r^0 + \lambda\sum_{n=1}^{\infty}\dot\epsilon_r^n$$

$$\dot\epsilon_\phi = \dot\epsilon_\phi^0 + \lambda\sum_{n=1}^{\infty}\dot\epsilon_\phi^n$$

$$\dot\epsilon_{r\phi} = \lambda\sum_{n=1}^{\infty}\dot\epsilon_{r\phi}^n \tag{15}$$

where $\dot\epsilon_r^0$, $\dot\epsilon_\phi^0$, $\dot\epsilon_r^n$, and $\dot\epsilon_\phi^n$ can be obtained easily. Substituting the "additional strains" $\dot\epsilon_r^n$, $\dot\epsilon_{r\phi}^n$ into the compatibility equation (2) making use of the stress function, the compatibility for the "additional stress functions" Φ_n reduces to the following equation:

$$\rho^4\frac{\partial^4\Phi_n}{\partial\rho^4} + b_1\rho^3\frac{\partial^3\Phi_n}{\partial\rho^3} + b_2\rho^2\frac{\partial^2\Phi_n}{\partial\rho^2} - b_2\rho\frac{\partial\Phi_n}{\partial\rho} + b_3\rho^2\frac{\partial^4\Phi_n}{\partial\rho^2\partial\phi^2} + b_4\rho\frac{\partial^3\Phi_n}{\partial\rho\partial\phi^2} + b_5\frac{\partial^2\Phi_n}{\partial\phi^2} + \frac{\partial^4\Phi_n}{\partial\phi^4} = T_n(\rho)\cos n\phi \tag{16}$$

where for $\theta_n(r) = C_n\rho^n + C_{-n}\rho^{-n}$

$$b_1 = 4K_1 - 2, \quad b_2 = 4K_1^2 - 8K_1 + 3, \quad b_3 = \frac{4}{m} - 2$$

$$b_4 = 8\frac{K_1}{m} + 6 - 4K_1 - \frac{20}{m},$$

$$b_5 = -4K_1^2 + 12K_1 - 8 - \frac{8K_1}{m} + \frac{16}{m} \tag{17}$$

and

$$T(\rho) = cK_1K_2\frac{8}{m}[(2n-2n^2)C_n b^{n+2}\rho^{n+2-2K_1} - (2n+2n^2)C_{-n}b^{(-n+2)}\rho^{-n+2-2K_1}] \tag{18}$$

C_n and C_{-n} are the general constants in $\theta_n(r)$ in Eq. (1). Equation (16) can be solved by assuming a stress function of the following form

$$\Phi_n(\rho,\phi) = R_n(\rho)\cos n\phi \tag{19}$$

Upon substitution of Eq. (19) into Eq. (16) an ordinary differential equation of the Euler type will be obtained as follows

$$\rho^4\frac{d^4R_n}{d\rho^4} + b_1\rho^3\frac{d^3R_n}{d\rho^3} + (b_2 - b_3n^2)\rho^2\frac{d^2R_n}{d\rho^2} - (b_2+n^2b_4)\rho\frac{dR_n}{d\rho} + (n^4 - b_5n^2)R_n = T_n(\rho) \tag{20}$$

The general and particular solutions of this differential equation for $R_1, R_2, \ldots R_n$ can be easily obtained, which upon substitution into Eq. (19) result in a complete solution for the "additional stress functions." The "additional stresses" can now be calculated from Eq. (16) which yield

$$\sigma_r^n = (\cos n\phi/b^2)\{C_1^n\rho^{s_1^n-2}(s_1^n-n^2) + C_2^n\rho^{s_2^n-2}(s_2^n-n^2) + C_3^n\rho^{s_3^n-2}(s_3^n-n^2) + C_4^n\rho^{s_4^n-2}(s_4^n-n^2) + D_1\rho^{L_1-2}(L_1-n^2) + D_2\rho^{L_2-2}(L_2-n^2)\} \tag{21a}$$

$$\tau_{r\phi}^n = (n\sin n\phi/b^2)\{C_1^n\rho^{s_1^n-2}(s_1^n-1) + C_2^n\rho^{s_2^n-2}(s_2^n-1) + C_3^n\rho^{s_3^n-2}(s_3^n-1) + C_4^n\rho^{s_4^n-2}(s_4^n-1) + D_1\rho^{L_1-2}(L_1-1) + D_2\rho^{L_2-2}(L_2-1)\} \tag{21b}$$

$$\sigma_\phi^n = (\cos n\phi/b^2)\{C_1^n\rho^{s_1^n-2}s_1^n(S_1^n-1) + C_2^n\rho^{s_2^n-2}s_2^n(s_2^n-1) + C_3^n\rho^{s_3^n-2}s_3^n(s_3^n-1) + C_4^n\rho^{s_4^n-2}s_4^n(s_4^n-1) + D_1\rho^{L_1-2}L_1(L_1-1) + D_2\rho^{L_2-2}L_2(L_2-1)\} \tag{21c}$$

where s_1^n through s_4^n are the solutions of the characteristic equation of Eq. (20); $L_1 = n+2-2K_1$, $L_2 = -n+2-2K_1$, and D_1 and D_2 are the constants in particular solution of Eq. (20); and C_1^n through C_4^n are the constants of integration in the general solution of Eq. (20).

Since the nonhomogeneous boundary conditions, $\sigma_r = -p$ at the inner surface of the cylinder, are already satisfied by the axisymmetric solution, the additional stresses should satisfy the homogeneous boundary conditions as stated below:

$$\sigma_r^n\left(\frac{1}{\beta},\phi\right) = \sigma_r^n(1,\phi) = \tau_{r\phi}^n\left(\frac{1}{\beta},\phi\right) = \tau_{r\phi}^n(1,\phi) = 0 \tag{22}$$

In applying the above boundary conditions, a set of four equations will be obtained which may be solved for the constants of integrations C_n^1 through C_n^4.

Conclusion

The theoretical derivation presented in this Note is a general method which can be applied to the problems of creep and plasticity or any other nonlinear stress analysis problem. The nature of the resulting differential equation after linearization depends on the constitutive relation between stress and strain, which is obtained using the experimental data. Any such constitutive law, although yielding a linear equation, may not result in an easily solved one. However, since the differential equation is always linear, a solution can always be obtained using classical procedures.

References

[1] Nadai, A. L., *Theory of Flow and Fracture of Solids*, Vols. 1 and 2, McGraw-Hill Book Co., New York, 1963.

[2] Rabotnov, Y. N., *Creep Problems in Structural Members*, John Wiley & Sons, New York, 1969.

[3] Eslami, M. R. and M. Sabbaghian, "Creep Relaxation of Nonaxisymmetric Thermal Stresses in Thick Walled Cylinders," *AIAA Journal*, Vol. 12, Dec. 1974, pp. 1652-1658.

[4] Kachanov, L. M., *The Theory of Creep*, National Lending Library, London, 1967.

[5] Rozenblyum, V. I., "Axisymmetric Creep of Cylindrical Bodies with Temperature Variation Along the Axis," PMTF, No. 6, 1961.

[6] Danyushevskii, A. and G. Listrinskii, "Creep in a Nonuniformly Heated Thick-Walled Tube Subjected to Internal Pressure," *Mechanics of Solids*, Vol. 1, No. 2, 1966, pp. 72-73.